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# ESTABLISHMENT OF BOUNDARY CONDITIONS FOR PROBLEMS OF CONDUCTIVE-CONVECTIVE HEAT TRANSFER IN DISPERSE SYSTEMS

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*As a result of analysis of various boundary conditions for the elliptic-type heat conduction equation which describes a stationary thermal state of a filtered granular bed, it has been established that the first-kind condition used earlier in literature for the inlet to the bed is incorrect and responsible for the disturbance of the overall heat balance of the system. The necessity of using the Danckwerts conditions, which reflect the specificity of the input of a heat carrier into the bed and output from it in the presence of longitudinal conductive heat transfer, is shown.*

The characteristic feature of most technological processes proceeding in a filtered granular bed is the inconstancy of temperatures in the bulk of the bed. As a rule, the heat-carrier flux passing through a bed is heated or cooled through the apparatus walls or during liberation or absorption of heat inside the granular bed (adsorption, desorption, drying, processes of combustion of a solid fuel, cooling of a heat-releasing bed of fuel elements, etc.).

Under steady-state conditions, in the absence of heat liberation inside a bed, by neglecting the difference between the temperatures of phases, the thermal state of the bed will be described by the following equation [1]:

$$c_{\text{f}}\rho_{\text{f}}u\,\frac{\partial T}{\partial x}=\lambda_r\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+\lambda_x\frac{\partial^2 T}{\partial x^2}\,. \tag{1}$$

As a boundary condition at the inlet to the bed it is convectional [1–4] to use the first-kind condition:

$$x=0\,,\quad T=T^{\text{in}}\,. \tag{2}$$

The remaining boundary conditions have the form

$$r=0\,,\ \ \frac{\partial T}{\partial r}=0\,;\quad r=R\,,\ \ -\lambda_r\frac{\partial T}{\partial r}=K\,(T-T_0)\,; \tag{3}$$

$$x\to\infty\,,\quad T=T_0\,. \tag{4}$$

We note that the heat-transfer coefficient $K$ includes resistance to heat transfer in the wall zone.

In [1], the solution of system (1)–(4) is given which is often cited in the literature [2, 3] and which was obtained by the method of Laplace integral transformation, for the use of which the validity of quite unapparent additional condition

$$x=0\,,\quad \frac{\partial T}{\partial x}=0 \tag{5}$$

was assumed. An analysis has shown that the solution given in [1] does not satisfy the differential equation (1) and boundary conditions (2) and (5). This seems to be due to the incorrect use of the Laplace integral transformation for

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Eq. (1), which is an elliptic-type equation and which describes a steady (stationary) thermal state of a granular bed, whereas the Laplace integral transformation, as is known [5], is applicable only to evolution-type equations which describe nonstationary processes, i.e., to parabolic- and hyperbolic-type equations.*

With allowance for the foregoing, system (1)–(4) was solved by the method of Hankel integral transformation with respect to the variable $r$ [7]. In a dimensionless form the system can be written as follows:

$$\frac{\partial \theta}{\partial x'} = \frac{\partial^2 \theta}{\partial (r')^2} + \frac{1}{r'} \frac{\partial \theta}{\partial r'} + \frac{1}{\mathrm{Pe}^2} \frac{\partial^2 \theta}{\partial (x')^2} , \tag{6}$$

with the boundary conditions

$$r' = 0 , \quad \frac{\partial \theta}{\partial r'} = 0 ; \quad r' = 1 , \quad \mathrm{Bi}\, \theta + \frac{\partial \theta}{\partial r'} = 0 ; \tag{7}$$

$$x' = 0 , \quad \theta = 1 ; \tag{8}$$

$$x' \to \infty , \quad \theta = 0 . \tag{9}$$

The solution of (6)–(9) has the form

$$\theta = \sum_{n=1}^{\infty} \frac{2\mathrm{Bi}}{\mathrm{Bi}^2 + \mu_n^2} \frac{J_0 (\mu_n r') \exp (s_{1n} x')}{J_0 (\mu_n)} , \tag{10}$$

where $\mu_n$ are the roots of the characteristic equation $\frac{J_0(\mu_n)}{J_1(\mu_n)} = \frac{\mu_n}{\mathrm{Bi}}$; $s_{1n} = \frac{\mathrm{Pe}^2}{2} - \frac{\mathrm{Pe}^2}{2} \sqrt{1 + 4 \frac{\mu_n^2}{\mathrm{Pe}^2}}$ . We note that for Pe $\to \infty$ Eq. (10) is reduced to the well-known solution of the parabolic heat-conduction equation [7]:

$$\theta = \sum_{n=1}^{\infty} \frac{2\mathrm{Bi}}{\mathrm{Bi}^2 + \mu_n^2} \frac{J_0 (\mu_n r') \exp (- \mu_n^2 x')}{J_0 (\mu_n)} . \tag{11}$$

The solution of Eq. (10) satisfies the differential equation (6) and boundary conditions (7)–(9).

The dimensionless heat flux $Q'$ removed from a portion of length $x'$ is determined, subject to (10), from the equation

$$Q' = \frac{Q}{\pi R^2 c_{\mathrm{f}} \rho_{\mathrm{f}} u \,(T^{\mathrm{in}} - T_0)} = \sum_{n=1}^{\infty} \frac{4\mathrm{Bi}^2}{\mathrm{Bi}^2 + \mu_n^2} \frac{\exp (s_{1n} x') - 1}{s_{1n}} . \tag{12}$$

Figure 1 shows the dependence of $Q'$ on $x'$. It is seen that with increase in $x'$ the limiting value of the quantity $Q'$ is not constant but is rather dependent on the Pe number. At the same time, from the heat balance condition for the total heat flux removed from the side surface of the tube for $x \to \infty$ we have

$$Q_\infty = \pi R^2 c_{\mathrm{f}} \rho_{\mathrm{f}} u \,(T^{\mathrm{in}} - T_0) . \tag{13}$$

*It should be noted that the simultaneous use of two boundary conditions (2) and (5) for elliptic-type equation (1) is inadmissible [6].

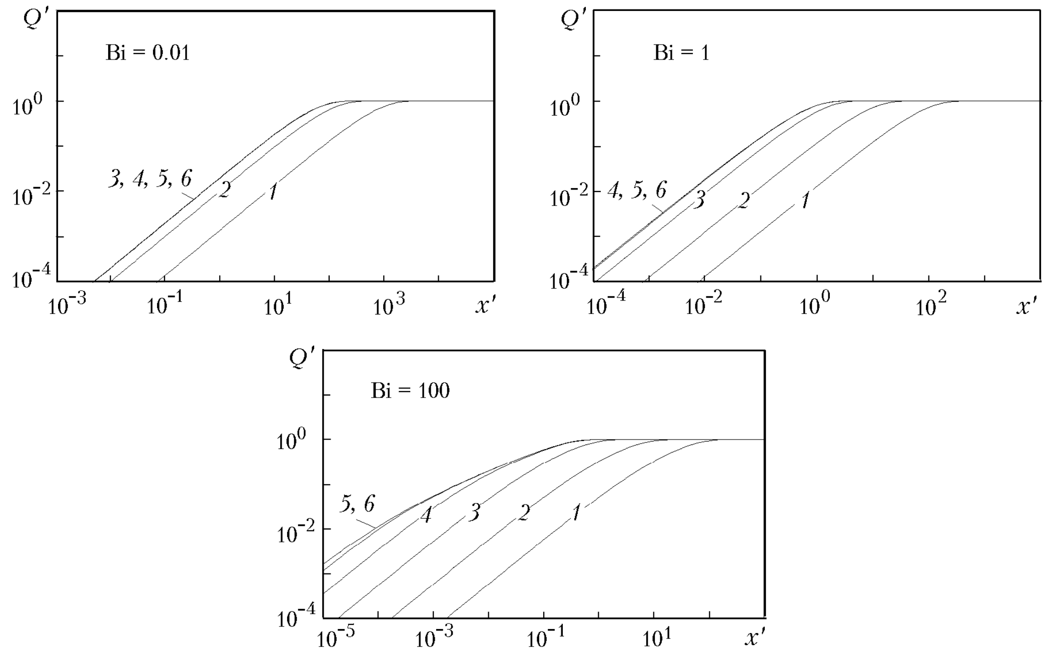


Fig. 1. Dependence of the dimensionless heat flux on $x'$ (calculation by Eq. (12)): 1) Pe = 0.01; 2) 0.1; 3) 1; 4) 10; 5) 100; 6) $\infty$.

It is evident that when $x' \to \infty$, $Q' \to 1$, which is not observed in the present case. This points to the disturbance of the overall heat balance in the system, when the process is described within the framework of model (1)–(4).

An analysis of the results obtained has allowed us to make an unambiguous conclusion on the incorrectness of using the condition $T|_{x=0} = T^{\text{in}}$, which does not reflect the balance of heat fluxes upon entry of the heat carrier into the bed in the presence of a longitudinal conductive heat transfer in it and, as a consequence, leads to the disturbance of the overall heat balance in the system.

For the density of longitudinal heat fluxes in the region adjacent to $x = 0$ we have

$$q_x\Big|_{x\to -0} = c_{\text{f}}\rho_{\text{f}} u T^{\text{in}} , \tag{14}$$

$$q_x\Big|_{x\to +0} = c_{\text{f}}\rho_{\text{f}} u T\Big|_{x=0} - \lambda_x \frac{\partial T}{\partial x}\Big|_{x=0} . \tag{15}$$

From the condition $q_x|_{x\to -0} = q_x|_{x\to +0}$ it follows that

$$T\Big|_{x=0} = T^{\text{in}} + \frac{\lambda_x}{c_{\text{f}}\rho_{\text{f}} u}\frac{\partial T}{\partial x}\Big|_{x=0} . \tag{16}$$

Relation (16), known in the literature as the first Danckwerts condition [8], leads to a temperature jump at $x = 0$ because of the presence of longitudinal heat conduction and shows that $T|_{x=0} \neq T^{\text{in}}$. Dependence (16) incorrectly reflects the condition of entry of a heat carrier into a granular bed, and it must be used in formulating the corresponding boundary-value problems. At the same time, the condition $T|_{x=0} = T^{\text{in}}$, as is seen, does not reflect the equality of the fluxes $q_x|_{x\to -0}$ and $q_x|_{x\to +0}$ and leads to disturbance of the heat balance in the system. We will write the correct boundary condition (16) in dimensionless form:

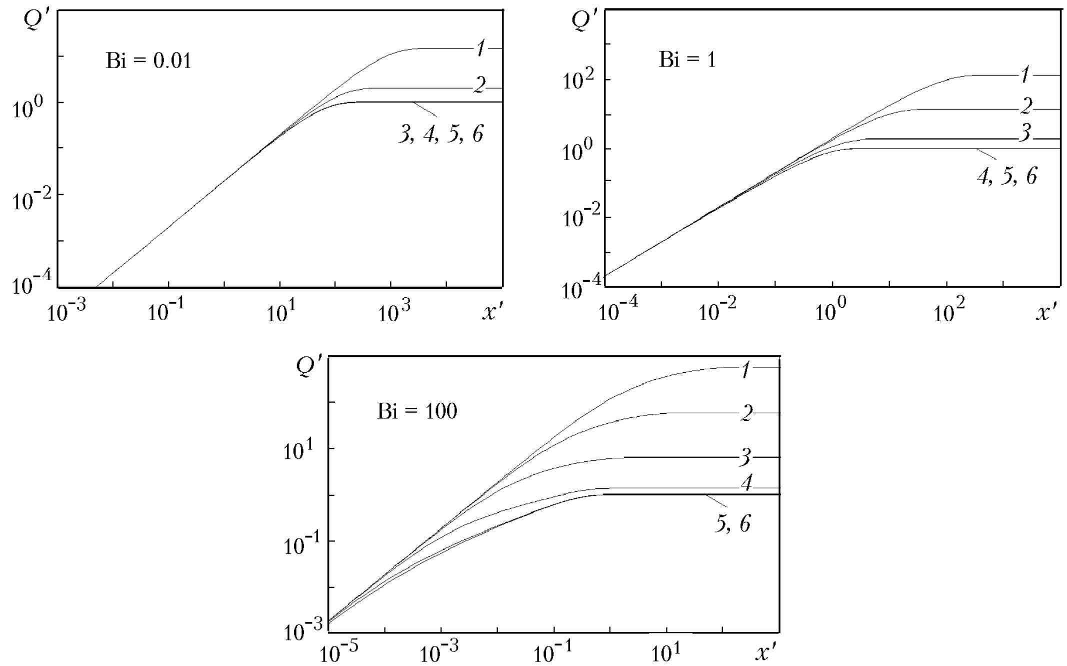


Fig. 2. Dependence of the dimensionless heat flux on $x'$ (calculation by Eq. (19)). Designations of 1–6 are the same as in Fig. 1.

$$x' = 0\,, \quad \theta = 1 + \frac{1}{\mathrm{Pe}^2}\frac{\partial\theta}{\partial x'}\,. \tag{17}$$

The solution of (6) with boundary conditions (7), (9), and (17) was obtained by the method of Hankel integral transformation with respect to the variable $r'$ [7]:

$$\theta = \sum_{n=1}^{\infty} \frac{2\mathrm{Bi}}{\mathrm{Bi}^2 + \mu_n^2} \frac{J_0(\mu_n r') \exp(s_{1n} x')\,\mathrm{Pe}^2}{J_0(\mu_n)\, s_{2n}}\,, \tag{18}$$

where $s_{2n} = \dfrac{\mathrm{Pe}^2}{2} + \dfrac{\mathrm{Pe}^2}{2}\sqrt{1 + 4\dfrac{\mu_n^2}{\mathrm{Pe}^2}}\,.$

The dimensionless heat flux $Q'$ is determined, with allowance for (18), from the equation

$$Q' = \frac{Q}{\pi R^2 c_{\mathrm{f}} \rho_{\mathrm{f}} u\,(T^{\mathrm{in}} - T_0)} = \sum_{n=1}^{\infty} \frac{4\mathrm{Bi}^2}{\mathrm{Bi}^2 + \mu_n^2} \frac{1 - \exp(s_{1n} x')}{\mu_n^2}\,. \tag{19}$$

From Fig. 2 it is seen that the magnitude of the dimensionless heat flux $Q'$ removed from the heat exchanger is independent of Pe when $x' \to \infty$ and is equal to unity. With allowance for the results presented in Fig. 1 we arrive at an important conclusion, in the context of the present work, on the necessity of using the Danckwerts condition (16), which correctly reflects the specifics of entry of a heat carrier into a granular bed in the presence of a longitudinal conductive heat transfer in it.

We note that in the case of a system of finite dimensions the following condition should be used at the exit from the bed:

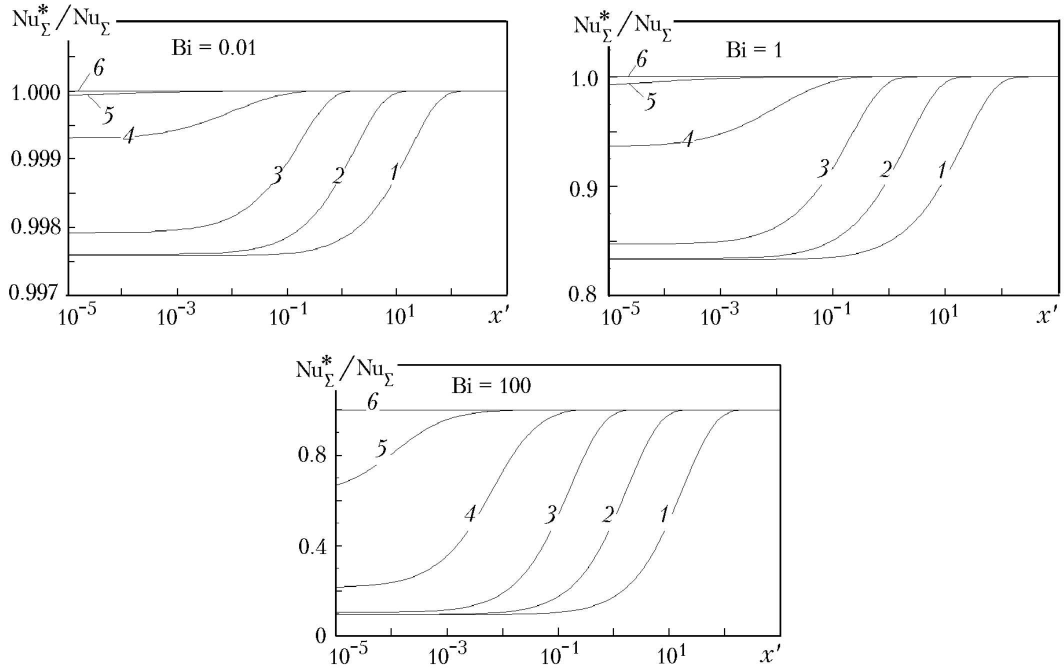


Fig. 3. Dependence of the ratio of dimensionless heat-transfer coefficients on $x'$. The designations of 1–6 are the same as in Fig. 1.

$$x = L\,,\quad \frac{\partial T}{\partial x} = 0\,, \tag{20}$$

which is the second Danckwerts condition [8]. It shows that the total heat flux leaving the bed is equal to the convective flux. It was shown in [9] that conditions (16) and (20) are applicable also in the case of a two-temperature model of a granular bed.

Of practical interest is a comparison of the heat-transfer coefficients determined without account for the Danckwerts boundary condition and when using it on the basis of Eqs. (10) and (18).

For the heat-transfer coefficient defined by the relation

$$K_{\Sigma} = K\,\frac{T|_{r'=1} - T_0}{\langle T\rangle - T_0}\,, \tag{21}$$

from Eq. (10) with allowance for the equation [7]

$$\int_0^1 J_0\,(\mu_n r')\, r' dr' = \frac{1}{\mu_n}\, J_1\,(\mu_n) \tag{22}$$

we obtain the dependence

$$K_{\Sigma} = K\,\frac{\displaystyle\sum_{n=1}^{\infty} \frac{\exp\,(s_{1n}x')}{(\mathrm{Bi}^2 + \mu_n^2)}}{\displaystyle 2\sum_{n=1}^{\infty} \frac{\mathrm{Bi}\exp\,(s_{1n}x')}{\mu_n^2\,(\mathrm{Bi}^2 + \mu_n^2)}}\,, \tag{23}$$

which in dimensionless form is written as

$$\mathrm{Nu}_{\Sigma} = \frac{\sum\limits_{n=1}^{\infty} \frac{\exp(s_{1n} x')}{(\mathrm{Bi}^2 + \mu_n^2)}}{2 \sum\limits_{n=1}^{\infty} \frac{\exp(s_{1n} x')}{\mu_n^2 (\mathrm{Bi}^2 + \mu_n^2)}} . \tag{24}$$

Similarly, for the heat-transfer coefficient determined from (21), subject to Eq. (18), we have

$$\mathrm{Nu}^{*}_{\Sigma} = \frac{\sum\limits_{n=1}^{\infty} \frac{\exp(s_{1n} x')}{(\mathrm{Bi}^2 + \mu_n^2)\, s_{2n}}}{2 \sum\limits_{n=1}^{\infty} \frac{\exp(s_{1n} x')}{\mu_n^2 (\mathrm{Bi}^2 + \mu_n^2)\, s_{2n}}} . \tag{25}$$

Graphs of the dependence of $\mathrm{Nu}^{*}_{\Sigma}/\mathrm{Nu}_{\Sigma}$ on $x'$ at different values of Bi and Pe are presented in Fig. 3. It is seen that as Pe increases, $\mathrm{Nu}^{*}_{\Sigma}/\mathrm{Nu}_{\Sigma} \to 1$. However, at low values of Pe and high values of Bi the difference between the heat-transfer coefficients calculated with account for different boundary conditions can be significant. It is evident that in this region, calculation of the heat-transfer coefficient should involve the Danckwerts condition.

## CONCLUSIONS

1. It has been established that the use of the first-kind boundary condition at the inlet to a bed is incorrect for the elliptic-type heat conduction equation, since it leads to disturbance of the overall heat balance of the system.
2. The necessity of using the Danckwerts conditions that reflect the specifics of entry of a heat carrier into a granular bed and exit from it in the presence of longitudinal heat conduction is shown.
3. It has been established that the heat-transfer coefficient calculated without allowance for the Danckwerts condition in the region of low coefficients of longitudinal heat conduction differs little from the heat-transfer coefficient calculated with account for the Danckwerts condition (Fig. 3).

## NOTATION

Bi = $KR/\lambda_r$, Biot number; $c_f$, heat capacity of a gas (liquid), J/(kg·K); $J_0$ and $J_1$, Bessel functions of first kind and zero and first orders; $K$, heat-transfer coefficient accounting for the thermal resistance of the wall zone and tube wall, W/(m$^2$·K); $L$, heat-exchanger length, m; $\mathrm{Nu}_{\Sigma} = K_{\Sigma}R/\lambda_r$, $\mathrm{Nu}^{*}_{\Sigma} = K^{*}_{\Sigma}R/\lambda_r$, Nusselt numbers; Pe = $Rc_f\rho_f u/(\lambda_x\lambda_r)^{1/2}$, Peclet number; $Q = 2\pi R \int_0^x q_r dx$, heat flux removed from a portion of length $x$, W; $Q' = Q/Q_{\infty}$; $q_r = -\lambda_r \frac{\partial T}{\partial r}$, density of radial heat flux, W/m$^2$; $r$, radial coordinate, m; $r' = r/R$; $R$, inner radius of heat exchanger, m; $T$, temperature, K; $T^{\mathrm{in}}$, temperature of the gas (liquid) at the inlet to the tube, K; $\langle T \rangle$, temperature average over the section $x$ = const, K; $T_0$, environmental temperature, K; $u$, rate of gas (liquid) filtration, m/sec; $x$, longitudinal coordinate, m; $x' = \lambda_r x/(R^2 c_f \rho_f u)$; $\theta = (T - T_0)/(T^{\mathrm{in}} - T_0)$, dimensionless relative temperature; $\lambda_x$ and $\lambda_r$, coefficients of longitudinal and transverse thermal conductivity of the granular bed, W/(m·K); $\rho_f$, density of gas (liquid), kg/m$^3$. Subscripts and superscripts: f, fluid (gas or liquid); in, inlet; Σ , total.

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